SPb SU Contest: LVII SPb SU Championship

August 27, 2023

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  1. A dynamic programming `dp[n][k]`: the minimum number of swaps, if we have considered the first n characters of the input string and typed k of them into the string "`spbsu`".
  2. A greedy algorithm that tries all possibilities for the central letter "`b`" and then selects other letters of the string "`spbsu`" to be as close to the chosen letter as possible.

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- Let g be the answer found by our algorithm. This number obviously cannot be smaller (in this context 0 is infinity) than the real answer, so we only have to prove that it is not greater or, equivalently, that g divides the length of each cycle.

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- Note that the intersection of any number of simple cycles is actually an intersection of some two of them.
- Any cycle could be represented as a XOR of several simple cycles, therefore, its length could be found as a linear combination with integer coefficients of lengths of those cycles and doubled lengths of their intersections. All summands are divisible by g, therefore, g divides the length of the cycle as well.

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- We can consider white and black squares independently, and we need to find a maximal matching in this graph.
- Note that each vertex in one part is connected to a segment of vertices in another.
- Finding a matching in a graph with this property is easy: we can simply sort all segments by their right ends and greedily choose the leftmost suitable vertex for each of them.

Bishops

- Now, the explicit construction.
- Rotate the board horizontally and then place bishops in cells with the following priority.
    1. On the left edge of the board.
    2. On the right edge of the board.
    3. On the one or two (depending on the parity of the board dimensions) middle rows going from the left to the right.

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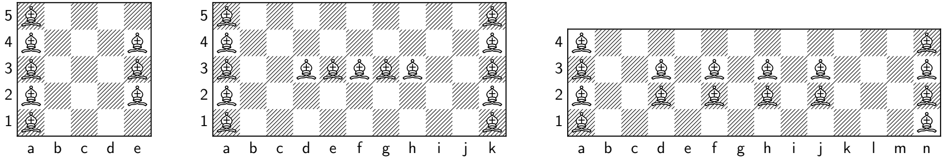
- Some examples:



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- The first solution. Naive.
- Arrange vertices into two types: light and heavy, depending on their degree in the original graph.
- For each heavy vertex, we will explicitly store all its neighbors of its color and the opposite color separately. To maintain this, when changing the color of any vertex, we will need to make at most $\mathcal{O}(\sqrt{m})$ additions and deletions, and at most one swap of the lists (if the recolored vertex is heavy).

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- We solved our problem with edges that have at least one endpoint in a heavy vertex, but there are still edges passing between light vertices.
- They can all be added to one large lazy queue for light edges since any recoloring of a light vertex will affect no more than $\mathcal{O}\left(\sqrt{m}\right)$ edges.

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- The only problem is the number of heavy vertices, which we solve by maintaining vertices with non-empty opposite-color neighbour lists in a lazy queue.
- The running time of both solutions is $\mathcal{O}\left(m + q\sqrt{m}\right)$ (assuming $n \leq m$ and $q > 0$).

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- The challenging part is avoiding Memory Limit Exceeded. To do so, we need to compress the incidence matrix.
- One way of doing this is to store it as a list of pairs of an index and a mask value.
- The complexity is $\mathcal{O}(m/w)$ per query. Some secondary optimizations may be required to get Accepted.

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- Split the array into blocks of size B (will be defined later), compute prefix and suffix sums inside each block and construct the sparse table with $k - 2$ operations on the blocks themselves.
- Now, we can answer any query that is not contained inside a single block.
- To answer any other query we need to construct the same sparse table recursively inside each block.

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- It could be shown that this construction is optimal both in time and memory.
- In case we know that the number of queries is $\Theta(n)$, we can achieve complexity $\Theta(n\alpha(n))$ where α is the inverse Ackermann function (and this is also optimal).

Operations per query	Time to construct (Θ omitted)
0	n^2
1	$n \log n$
2	$n \log \log n$
3	$n \log^* n$
4	$n \log^* n$
5	$n \log^{**} n$
6	$n \log^{**} n$
7	$n \log^{***} n$
$\vdots$	$\vdots$
k + 1	$n \log^{\overbrace{* * \cdots *}^{\lfloor k/2 \rfloor}} n$
$\vdots$	$\vdots$